

Symplectic quasi-states on the quadric surface and Lagrangian submanifolds

Yakov Eliashberg and Leonid Polterovich

June 12, 2010

Abstract

The quantum homology of the monotone complex quadric surface splits into the sum of two fields. We outline a proof of the following statement: The unities of these fields give rise to distinct symplectic quasi-states defined by asymptotic spectral invariants. In fact, these quasi-states turn out to be “supported” on disjoint Lagrangian submanifolds. Our method involves a spectral sequence which starts at homology of the loop space of the 2-sphere and whose higher differentials are computed via symplectic field theory, in particular with the help of the Bourgeois-Oancea exact sequence.

1 Introduction and main result

The quantum homology of the monotone complex quadric surface $S^2 \times S^2$ splits into the sum of two fields. The unities of these fields give rise to symplectic quasi-states defined by asymptotic spectral invariants (see [11]). One of these quasi-states is “supported” on a Lagrangian sphere, the anti-diagonal [12]. Our main finding (see Theorem 1.1 below) is that the second quasi-state is “supported” on the exotic monotone Lagrangian torus described in [12] which is disjoint from the anti-diagonal. Thus these quasi-states are distinct, and the exotic torus has strong symplectic rigidity properties. Let us pass to precise formulations.

Let (W, ω) be the standard symplectic quadric surface $S^2 \times S^2$, where both S^2 factors have equal areas 1. Consider the field $\mathcal{K} = \mathbb{C}[[t^{-1}, t]]$ of Laurent

series and the quantum homology algebra $QH(W, \omega) = H_*(W, \mathbb{C}) \otimes \mathcal{K}$ which is graded by

$$\deg(at^N) = \deg(a) + 4N .$$

The graded component QH_4 is an algebra over \mathbb{C} with respect to the quantum product. It splits into the sum of two fields generated by idempotents $e_{\pm} = (1 \pm Pt)/2$, where $1 = [W]$ is the fundamental class and P stands for the class of the point. Define the functionals $\zeta_{\pm} : C^{\infty}(W) \rightarrow \mathbb{R}$ by

$$\zeta_{\pm}(H) = \lim_{E \rightarrow \infty} c(e_{\pm}, EH)/E ,$$

where $c(\cdot, \cdot)$ is a spectral invariant [17, 16].

It was shown in [11] that these functionals are *symplectic quasi-states*, that is they are monotone, linear on all Poisson-commutative subspaces and normalized by $\zeta_{\pm}(1) = 1$. Recall that a *quasi-measure* associated to a quasi-state ζ is a set-function whose value on a closed subset X equals, roughly speaking, $\zeta(\chi_X)$ where χ_X is the indicator function of X . We write τ_{\pm} for the quasi-measures associated to ζ_{\pm} . The reader is referred to [10, 11] for further preliminaries.

We view W as the symplectic cut [14] of the unit cotangent bundle $\{|p| \leq 1\} \subset T^*S^2$, where the zero section is identified with the Lagrangian anti-diagonal $L \subset W$, and the length $|p|$ is understood with respect to the standard spherical metric so that the length of the equator equals 1. The level $\{|p| = 1/2\}$ contains unique (up to a Hamiltonian isotopy) monotone Lagrangian torus denoted by K . Write $T \subset W$ for the Lagrangian Clifford torus (the product of equators).

It has been proved in [12, Example 1.20] that $\tau_-(L) = 1$ and $\tau_-(K) = 0$. Together with the equality $\tau_-(T) = \tau_+(T) = 1$ (see [11]) this yields that K and T are not Hamiltonian isotopic.

Theorem 1.1. $\tau_+(L) = 0, \tau_+(K) = 1$. *In particular, $\zeta_- \neq \zeta_+$.*

Equality $\tau_+(K) = 1$ yields that K is e_+ -superheavy in the terminology of [12]. In particular, it is non-displaceable and intersects every image of the Clifford torus under a symplectomorphism.

The rest of the note contains a detailed outline of the proof of Theorem 1.1. Our method involves a spectral sequence which starts at homology of the loop space of the 2-sphere and whose higher differentials are computed via

symplectic field theory. The main technical ingredient comes from a paper by Bourgeois and Oancea [5].

Non-displaceability of the exotic torus K has been recently established via Lagrangian Floer homology by several independent groups of researchers: Fukaya, Oh, Ohta and Ono [13], Chekanov and Schlenk [6] and Wehrheim (unpublished). The paper [6] presents new constructions of exotic Lagrangian tori in the product of spheres. A related construction of exotic tori is given by Biran and Cornea in [4] in the context of their study of narrow Lagrangian submanifolds. It would be interesting to understand whether these tori can be distinguished by appropriate symplectic quasi-measures.

The paper [13] contains a more general version of Theorem 1.1. According to [13], the exotic torus K lies in an infinite family of non-displaceable Lagrangian tori whose Liouville class varies. Though our approach is quite different from the one in [13], there are some similarities which deserve further exploration. Let us mention also that the tori of the above-mentioned family appear as invariant sets of a semitoric integrable system (see [18] and Section 2 below). In dimension four, semitoric means that one of the integrals generates a Hamiltonian circle action. The study of this class of integrable systems was initiated recently in [18]. Semitoric systems have some amusing properties and appear in meaningful physical models. It would be interesting to detect “symplectically rigid” invariant Lagrangian tori in other examples of semitoric systems.

2 Reduction to a Floer-homological calculation

We work with Floer homology with \mathbb{C} -coefficients. In our conventions on the Conley-Zehnder indices, the PSS-isomorphism identifies FH_k with QH_{k+2} (see [16] for preliminaries).

Throughout that paper, we denote by Σ the diagonal in $W = S^2 \times S^2$. In our picture, Σ is obtained from the hypersurface $\{|p| = 1\}$ by the symplectic cut [14].

Fix $r \in (0; 1/2)$, $E > 0$ large enough, and $\epsilon > 0$ small enough. We assume that the data is “non-resonant”: $1/r \notin \mathbb{Z}$ and $(E + \epsilon)/\epsilon \notin \mathbb{Z}$.

Consider a Hamiltonian $H_E(|p|)$ on W given by a piece-wise linear function which equals E on $U := \{|p| \leq r - \epsilon\}$ and equals $-\epsilon$ on $V := \{|p| \geq r\}$.

We refer e.g. to [9] for a discussion on Floer-homological calculations with piece-wise linear Hamiltonians.

Orbits of period 1 of H_E form several critical submanifolds, which we are going to describe now. Each of these submanifolds is equipped with a Morse function which is used for a small perturbation of the action functional associated to H_E . In addition, we fix capping discs for orbits from these submanifolds. Critical points of these Morse functions together with the capping discs give rise to generators of the Floer complex. Let us pass to the precise description of this data.

THE MAXIMUM SET U : Here we have constant orbits capped with the constant discs. We choose an exhausting Morse function f_U on U with two critical points: a saddle point x_0 of Morse index 2 and a maximum point x_2 . Their Conley-Zehnder indices are 0 and 2 respectively, and their actions equal E . We refer to the elements of the Floer complex of the form γt^{-N} for $\gamma \in \{x_0, x_2\}$ as **U -generators**, and call the number N *the t -degree*.

THE MINIMUM SET V : Here we have constant orbits capped by the constant discs. We choose an exhausting Morse function f_V on V with two critical points: a saddle point y_0 of Morse index 2 and a minimum point y_{-2} . Their Conley-Zehnder indices are 0 and -2 respectively, and their actions equal $-\epsilon$. We refer to the elements of the Floer complex of the form γt^{-N} for $\gamma \in \{y_0, y_{-2}\}$ as **V -generators**.

NON-CONSTANT ORBITS: They form two series of submanifolds diffeomorphic to $\mathbb{R}P^3$. We denote these submanifolds by Z_k^\pm . Here the lower index k corresponds to k -times-covered simple closed geodesics on L , Z^+ stands for the orbits on the submanifold $\{|p| = r - \epsilon\}$ and Z^- stands for the orbits on the submanifold $\{|p| = r\}$. Note that the multiplicity k satisfies inequalities $k \geq 1$ and

$$k \leq \frac{E + \epsilon}{\epsilon}. \quad (1)$$

In the discussion below we assume that $k \geq 1$ is arbitrary, and that inequality (1) is an extra restriction which selects orbits relevant for the Floer complex corresponding to the fixed value of $\epsilon > 0$. Next, we fix a Morse function f_k^\pm on Z_k^\pm with critical points \check{m}_k^\pm , \hat{m}_k^\pm , \check{M}_k^\pm and \hat{M}_k^\pm of Morse indices 0, 1, 2, 3 respectively. It will be convenient for us to choose f_k in such a way that the orbits in each pair of orbits $(\hat{m}_k^\pm, \check{m}_k^\pm)$ and $(\hat{M}_k^\pm, \check{M}_k^\pm)$ represent the same unparameterized orbit. The orbits from Z_k^\pm are capped by discs lying in $W \setminus \Sigma$. We refer to the elements of the Floer complex of the form γt^{-N} for

$\gamma \in \check{m}_k^+, \hat{m}_k^+, \check{M}_k^+, \hat{M}_k^+$ as **upper generators** and for $\gamma \in \check{m}_k^-, \hat{m}_k^-, \check{M}_k^-, \hat{M}_k^-$ as **lower generators**, and call the number N the t -degree.

The Conley-Zehnder indices of the generators corresponding to the non-constant orbits are given in Table 1 below.

Table 1: Indices of upper/lower generators

	\check{m}_k	\hat{m}_k	\check{M}_k	\hat{M}_k
upper	$2k - 1$	$2k$	$2k + 1$	$2k + 2$
lower	$2k - 2$	$2k - 1$	$2k$	$2k + 1$

The actions of the lower generators equal $-\epsilon + kr$ and of the upper generators $E + k(r - \epsilon)$.

In what follows we write $CZ(\gamma)$ for the Conley-Zehnder index of a capped orbit γ and $A(\gamma)$ for its action. We have

$$CZ(\gamma t^{-N}) = CZ(\gamma) - 4N, \quad A(\gamma t^{-N}) = A(\gamma) - N.$$

A direct calculation (crucially based on the fact that $r < 1/2$) yields the following lemma which will be used throughout the paper.

Lemma 2.1. (i) All lower generators and V -generators of the Conley-Zehnder indices 1, 2, 3 have action < 1 .

(ii) All upper generators and U -generators of the Conley-Zehnder indices 1, 2, 3 have action $< E + 1$ and non-negative t -degree.

(iii) Let γ be an upper generator of the Conley-Zehnder index 1, 2, 3 and action $A(\gamma) > 1$. Then inequality (1) holds automatically: $k < \frac{E+\epsilon}{\epsilon}$.

(iv) There exist numbers $\mu_-(E) < \mu_+(E)$, $\mu_{\pm}(E) \rightarrow \infty$ as $E \rightarrow \infty$ with the following property: Let γ be an upper generator or a U -generator of the Conley-Zehnder index 1, 2, 3 and t -degree N . Then $A(\gamma) > 1$ for $N \leq \mu_-$ and $A(\gamma) < 1$ for $N > \mu_+(E)$.

Proof. LOWER GENERATORS: Take $\gamma_k \in \{\check{m}_k^-, \hat{m}_k^-, \check{M}_k^-, \hat{M}_k^-\}$ and put $\gamma = \gamma_k t^{-N}$. We have that

$$CZ(\gamma) = 2k + j - 4N \in [1; 3], \quad j = -2, -1, 0, 1.$$

Thus $0 \leq 2k - 4N \leq 4$. Note that

$$A(\gamma) = kr - \epsilon - N .$$

Since $r < 1/2$ we have that

$$A(\gamma) < (k/2 - N) - \epsilon < 1 .$$

UPPER GENERATORS: Take $\gamma_k \in \{\check{m}_k^+, \hat{m}_k^+, \check{M}_k^+, \hat{M}_k^+\}$ and put $\gamma = \gamma_k t^{-N}$. We have that

$$CZ(\gamma) = 2k + j - 4N \in [1; 3], \quad j = -1, 0, 1, 2 .$$

Thus

$$0 \leq 2k - 4N \leq 4 . \tag{2}$$

In particular, $N \geq 0$ since $k \geq 1$. Note that

$$A(\gamma) = E + kr - k\epsilon - N . \tag{3}$$

Since $r < 1/2$ we have that

$$A(\gamma) < E + (k/2 - N) - k\epsilon \leq E + 1 .$$

Put $\kappa = 1 - 2r + 2\epsilon$. Observe that by (2) and (3) $A(\gamma) > 1$ yields $k < 2E/\kappa < (E + \epsilon)/\epsilon$ which proves statement (iii) of the lemma.

Further, pick

$$\mu_-(E) < (E - 1)/\kappa, \mu_+(E) > (E - \kappa)/\kappa .$$

Statement (iv) of the lemma readily follows from (2) and (3).

Finally, the only U -generator of the index 1, 2, 3 is x_2 , and its action equals E and its t -degree equals 0. The only V -generator of index 2 is $y_{-2}t$ and its action equals $1 - \epsilon$. This completes the proof of statements (i) and (ii) of the lemma. \square

Lemma 2.2 (Main lemma). $HF_2^{(1;E+1)}(H_E) = \mathbb{C}$.

Proof of Theorem 1.1 modulo Main Lemma:

STEP 1: Look at the diagram

$$\begin{array}{ccccc}
 & & HF_3^{(E+1,+\infty)}(H_E) & & \\
 & & \downarrow j & & \\
 HF_2^{(-\infty,1)}(H_E) & \xrightarrow{k} & HF_2^{(-\infty,E+1)}(H_E) & \xrightarrow{l} & HF_2^{(1,E+1)}(H_E) = \mathbb{C} \\
 & \searrow m & \downarrow i & & \\
 & & HF_2^{(-\infty,\infty)} = QH_4(M) = \mathbb{C}^2 & &
 \end{array}$$

Here the horizontal and the vertical lines are exact, and the triangle is commutative. Since $e_{\pm} = (1 \pm Pt)/2$ and $\max H_E = E$ the spectral invariants $c(e_{\pm}, H_E)$ do not exceed $E + 1$. Thus, since QH_4 is generated by e_-, e_+ the map i is onto. By Lemma 2.1(i),(ii) $HF_3^{(E+1,+\infty)}(H_E) = 0$. This yields that $j = 0$ so i is an isomorphism, and in particular l has a non-trivial kernel. Thus $k \neq 0$ and we conclude that $m \neq 0$.

Assume that some non-zero quantum homology class $a = \alpha e_- + \beta e_+$, $\alpha, \beta \in \mathbb{C}$, lies in the image of m . This yields $c(a, H_E) \leq 1$. Since $\tau_-(L) = 1$ (see [12]), we have that $c(e_-, H_E) \geq E$, and therefore $\beta \neq 0$. Observe that the quantum product $e_+ * e_-$ equals 0, while $e_+ * e_+ = e_+$, $e_- * e_- = e_-$. Thus, by the triangle inequality for spectral invariants,

$$c(e_+, H_E) = c(a * e_+, H_E) \leq c(a, H_E) + c(e_+, 0) \leq 2.$$

Since this holds for every E and $r < 1/2$, we conclude that

$$\tau_+(\{|p| < 1/2\}) = 0. \quad (4)$$

STEP 2: Observe now that the Hamiltonian $|p|^2$ on $W \setminus \Sigma$ extends to a *smooth* Hamiltonian on the whole W . This Hamiltonian is integrable and yields a foliation of $W \setminus (L \cup \Sigma)$ by Lagrangian tori. Look at these tori in the tube $\{|p| \geq 1/2\}$. One readily checks by an argument in the spirit of [15], that all these tori besides the monotone exotic torus K are displaceable.

One can prove the displaceability directly in the following way. Write $W = S^2 \times S^2$ as

$$\{x_1^2 + y_1^2 + z_1^2 = 1\} \times \{x_2^2 + y_2^2 + z_2^2 = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

Introduce functions F and G on W by

$$\begin{aligned} F(x_1, y_1, z_1, x_2, y_2, z_2) &= z_1 + z_2 , \\ G(x_1, y_1, z_1, x_2, y_2, z_2) &= x_1 x_2 + y_1 y_2 + z_1 z_2 , \end{aligned}$$

and a map

$$\Phi = (F, G) : W \rightarrow \mathbb{R}^2 . \quad (5)$$

One can check directly that within this model the Hamiltonian $|p|$ corresponds to $\sqrt{(1+G)/2}$. It defines an integrable Hamiltonian system with an integral F (since F generates a circle action, such an integrable system is *semitoric* [18]). The above-mentioned Lagrangian tori are given by

$$N_{a,b} := \Phi^{-1}(a, b) = \{z_1 + z_2 = a, x_1 x_2 + y_1 y_2 + z_1 z_2 = b\} .$$

The monotone torus K is given by $N_{0,-1/2}$.

Note that for $a \neq 0$, $N_{a,b}$ is displaceable by

$$(x_1, y_1, z_1, x_2, y_2, z_2) \rightarrow (-x_1, y_1, -z_1, -x_2, y_2, -z_2) .$$

For $a = 0$, the torus $N_{0,b}$, $b \in (-1/2, 1)$ can be displaced inside the hypersurface $\Pi := \{z_1 + z_2 = 0\}$. Indeed, let ϕ_i be the polar angle in the (x_i, y_i) -plane. Consider a fibration $\tau : \Pi \rightarrow C := (-1; 1) \times S^1$ given by

$$(x_1, y_1, z_1, x_2, y_2, z_2) \rightarrow (z_1, \phi_1 - \phi_2) .$$

One readily checks that for every $(z, \theta) \in C$ the preimage $\tau^{-1}(z, \theta)$ consists of a closed orbit of the Hamiltonian $z_1 + z_2$. Thus for every simple closed curve $\alpha \subset C$, the preimage $\tau^{-1}(\alpha)$ is a Lagrangian torus in Π . Denote by σ the push-forward to C of the symplectic form restricted to Π . Since the symplectic form on W is given by

$$\frac{1}{4\pi}(dz_1 \wedge d\phi_1 + dz_2 \wedge d\phi_2) ,$$

we have that $\sigma = (4\pi)^{-1}dz \wedge d\theta$. Furthermore, $N_{0,b} = \tau^{-1}(\alpha_b)$ with

$$\alpha_b = \left\{ z^2 = \frac{\cos \theta - b}{\cos \theta + 1} \right\} .$$

Observe that α_b is a contractible simple closed curve in C . Integration yields

$$\frac{1}{4\pi} \int_{\alpha_{-1/2}} z d\theta = \frac{1}{2} = \frac{1}{2} \text{Area}_\sigma(C) .$$

For $b > -1/2$ the curve α_b lies inside the disc bounded by $\alpha_{-1/2}$ in C . Thus α_b is displaceable in C , and therefore, by lifting the displacing isotopy to Π , we get that $N_{0,b}$ is displaceable in Π . This completes the proof of the displaceability.

STEP 3: Consider the push-forward $\Phi_*\tau_+$ of the quasi-measure τ_+ to the plane \mathbb{R}^2 by the map Φ given by (5). Let (u, v) be Euclidean coordinates on \mathbb{R}^2 . Since F and G Poisson commute, $\Phi_*\tau_+$ extends to a measure, say σ on \mathbb{R}^2 . Recall that in our model of W the function $|p|$ corresponds to $\sqrt{(1+G)/2}$, and hence the tube $\{|p| < 1/2\}$ is given by $\{G < -1/2\}$. Formula (4) above implies that the support of σ lies in $\{v \geq -1/2\}$.

Further, by Step 2, every non-empty fiber $\Phi^{-1}(a, b)$ with

$$b \geq -1/2, (a, b) \neq (0, -1/2)$$

is displaceable in W . Recall [11] that every Floer-homological symplectic quasi-measure vanishes on displaceable subsets, and hence a point $(a, b) \in \mathbb{R}^2$ cannot lie in the support of σ provided the set $\Phi^{-1}(a, b)$ is displaceable. Therefore the support of σ consists of the single point $(0, -1/2)$, so that σ is the Dirac δ -measure concentrated in this point. Since the torus K is given by $\Phi^{-1}(0, -1/2)$, we get that

$$\tau_+(K) = \sigma((0, -1/2)) = 1.$$

This completes the proof of the theorem. \square

3 Proof of the Main Lemma

THE STRATEGY OF CALCULATION:

By Lemma 2.1 The Floer complex $CF_i^{(1;E+1)}(H_E)$, $i = 1, 2, 3$ is generated by upper generators and U -generators in the action window $(1; E+1)$ satisfying the multiplicity bound (1). The lower generators and V -generators leave the stage. *Thus we shall suppress the upper index + and set $Z_k = Z_k^+$, $f_k := f_k^+$, $\check{m}_k = \check{m}_k^+$, etc.*

Denote by B_n , $n \geq 0$ the span over \mathbb{C} of generators

$$\gamma \in \{x_0, x_2, \check{m}_k, \hat{m}_k, \check{M}_k, \hat{M}_k, k \geq 1\}$$

of the Conley-Zehnder index n .

Put $C_{i,s} = t^{-s}B_{i+4s}$, where $i \geq 0$ and $s \geq 0$. Write $C_i = \oplus_s C_{i,s}$ and $C = C_1 \oplus C_2 \oplus C_3$. Denote by $D \subset C$ the subspace consisting of the elements of action < 1 and set $D_{i,s} = D \cap C_{i,s}$. By Lemma 2.1(i)-(iii) the Floer complex of H_E in the action window $(1; E+1)$ and in degrees 1, 2, 3 is given by $(C/D, \delta)$, where $\delta : C/D \rightarrow C/D$ is the Floer differential. The differential δ has the form

$$\delta = \delta_0 + t^{-1}\delta_1 + t^{-2}\delta_2 + \dots \mod D \quad (6)$$

with $\delta_l : B_* \rightarrow B_{*+4l-1}$. Let us emphasize that only negative powers of t appear in this expression: this follows from the fact that Floer trajectories of H_E are holomorphic near Σ and intersect it positively. With this notation,

$$HF_2^{(1;E+1)}(H_E) = H_2(C/D, \delta). \quad (7)$$

To motivate the strategy of calculation of this homology group, identify

$$CF_i^{(1;E+1)}(H_E) = CF_i^{(1-E;1)}(H_E - E) \quad (8)$$

and look at $C_{i,0}/D_{i,0}$ considered as a subspace of $CF_i^{(1-E;1)}(H_E - E)$. With this identification homology of the complex $(\oplus C_{i,0}/D_{i,0}, \delta_0)$ converge to symplectic homology $SH(U')$ of the domain $U' = \{|p| < r\}$ in the action window $(-\infty; 1)$ provided $E \rightarrow \infty$ and $\epsilon \rightarrow 0$. Indeed, functions $H_E - E$ restricted to U' form an exhausting sequence used in the definition of symplectic homology, the complex $\oplus C_{i,0}/D_{i,0}$ is generated by closed orbits of $H_E - E$ capped inside U' and the differential δ_0 counts the Floer trajectories lying inside U' . The contribution of the lower generators disappears in this limit.

Now we can formulate the intuitive idea behind our calculation: The complex $(C/D, \delta)$ can be considered as a properly understood deformation of $(\oplus C_{i,0}/D_{i,0}, \delta_0)$ which involves capping discs and Floer trajectories intersecting Σ . Eventually, the required homology $H(C/D, \delta)$ can be calculated by an appropriate spectral sequence which starts at $SH(U')$.

To make this precise, we use the technology developed by Bourgeois and Oancea [5] (and extended further in [3]) who identified symplectic homology of the Liouville domain $\{|p| < r\}$ with the homology of the complex $B := \oplus B_i = \oplus_i C_{i,0}$ equipped with certain differential d_0 which will be described below. In fact we shall introduce an appropriate deformation of their construction which takes into account the fact that Floer cylinders can intersect Σ and which will enable us to calculate homology of the deformed complex $(C/D, \delta)$.

As a graded and filtered ¹ vector space the deformed Bourgeois-Oancea complex is described as follows. Introduce the ring Λ consisting of all Laurent series of the form

$$\sum_{s=0}^{+\infty} \lambda_s t^{-s}, \lambda_s \in \mathbb{C}.$$

With this notation the deformed Bourgeois-Oancea complex is given by $QB := B \otimes_{\mathbb{C}} \Lambda$. As before, this complex is graded by $CZ(\gamma t^{-s}) = CZ(\gamma) - 4s$ and filtered by the symplectic action of H_E : $A(\gamma t^{-s}) = A(\gamma) - s$. Its differential d is Λ -linear and has the form

$$d = d_0 + t^{-1}d_1 + t^{-2}d_2 + \dots,$$

with $d_l : B_* \rightarrow B_{*+4l-1}$. It is instructive to view (QB, d) as a “quantum” deformation of the complex (B, d_0) where t plays the role of a deformation parameter. By [5] the group $H(B, d_0)$ coincides with symplectic homology of the Liouville domain $\{|p| < r\} \subset T^*S^2$. The latter, according to [1, 20], equals homology of the free loop space of $S^2 = L$. Therefore

$$H(B, d_0) = H(\mathcal{LS}^2). \quad (9)$$

In order to describe the differential d we need some preliminaries.

STRETCHING-THE-NECK:

Denote by $\pi : \nu \rightarrow \Sigma$ the holomorphic normal line bundle to Σ in $W = \mathbb{CP}^1 \times \mathbb{CP}^1$. Perform a stretching-the-neck [8, 2] of W at the hypersurfaces $\{|p| = r - \epsilon\}$ and $\{|p| = r\}$. The manifold W splits into three pieces which after gluing in (asymptotically) cylindrical ends at their boundaries will be identified with $W_{left} := W \setminus \Sigma$, $W_{middle} := \nu \setminus \Sigma$ and $W_{right} := \nu$.

It would be convenient to view orbits from Z_k as k -times-covered unit circles of the bundle ν . In particular, the projection $\pi : \nu \rightarrow \Sigma$ gives rise to the natural map $\pi^k : Z_k \rightarrow \Sigma$.

We shall assume that the exhausting Morse function f_U is defined on the whole W_{left} .

MATRIX COEFFICIENT $(d_l \gamma_+, \gamma_-)$ FOR UPPER GENERATORS γ_+, γ_- :

Denote by $P_{a,b}$ the cylinder $\mathbb{C} \setminus 0$ with the set of *negative* interior punctures $a = \{a_1, \dots, a_{l_-}\}$ and the set of *positive* interior punctures $b = \{b_1, \dots, b_{l_+}\}$. Here we put $l_+ = l$ (recall that we are defining d_l).

¹ One should shift our filtration by E to get the standard filtration on symplectic homology used in [5], see (8).

Let γ be an orbit from Z_k . In our picture it is interpreted in two different ways. First, it is a point of the corresponding critical variety Z_k (recall that upper index $+$ is omitted). We denote this point by A_γ . Second, γ is a (parameterized, in general multiply covered) unit circle in the fiber of the bundle ν over the projection $\pi(A_\gamma)$.

In what follows we work with holomorphic maps

$$u : P_{a,b} \rightarrow W_{middle} = \nu \setminus \Sigma .$$

We say that u *enters* γ at a puncture $\eta \in a \cup \infty$ if $u(z)/|u(z)| \rightarrow \gamma(\arg(z))$ and $|u(z)| \rightarrow \infty$ as $z \rightarrow \eta$. We say that u *exits* γ at a puncture $\eta \in b \cup 0$ if $u(z)/|u(z)| \rightarrow \gamma(\arg(z))$ and $|u(z)| \rightarrow 0$ as $z \rightarrow \eta$.

Suppose that u exits α at 0, enters β at ∞ and in addition exits (resp. enters) *some* orbits at all positive (resp. negative) interior punctures. We shall denote this by

$$\alpha \xrightarrow{u} \beta .$$

We shall also book-keep the quantity l_- by putting

$$\text{weight}(u) = 2^{l_-} . \quad (10)$$

Note that geometrically such u 's either are contained in a single fiber of ν , or correspond to multi-sections of ν with zeroes at $\pi(A_\alpha)$ and the projections of the asymptotic images of the positive punctures, and with poles at $\pi(A_\beta)$ and the projections of the asymptotic images of the negative punctures.

Suppose now that α, β belong to the same critical manifold Z_k . We shall write

$$\alpha \rightsquigarrow^v \beta \quad (11)$$

if v is a parameterized piece of trajectory of the negative gradient flow of f_k joining the points A_α and A_β .²

Let γ_+, γ_- be two upper generators representing critical points of f_{k_+}, f_{k_-} on Z_{k_+}, Z_{k_-} respectively. Assume that $k_+ \neq k_-$. Consider all possible configurations of the form

$$\gamma_+ \rightsquigarrow^v \alpha \xrightarrow{u} \beta \rightsquigarrow^w \gamma_- . \quad (12)$$

²We will assume that the gradient vector field for each function f_k satisfies the following condition: the 1-dimensional stable manifold of \check{M}_k (resp. the unstable manifold of \hat{m}_k) consists of orbits which differ from \check{M}_k (resp. from \hat{m}_k) only by their parameterization.

We call $\text{weight}(u)$ the weight of this configuration. Note that the right and/or the left arrow could be empty. In case $k_+ = k_- = k$, we consider configurations of the form

$$\gamma_+ \overset{v}{\rightsquigarrow} \gamma_- , \quad (13)$$

and its weight is put to be 1. Finally, we define $(d_l \gamma_+, \gamma_-)$ as the sum of weights taken over the 0-dimensional components of the moduli spaces of configurations of the form (12) and (13). Note that in the definition of the moduli spaces (12) the markers a, b are varying as well. In addition, each weight should be taken with a sign responsible for the orientation of the moduli space. The orientation issue will be ignored in this note.

MATRIX COEFFICIENTS $(d_l x, \gamma)$ AND $(d_l \gamma, x)$ FOR AN UPPER ORBIT γ AND AN U -ORBIT x :

First, note that all the coefficients $(d_l \gamma, x)$ vanish for $l > 0$ by index reasons, so we do not need to describe here the algorithm for their computing. For the description of $(d_0 \gamma, x)$ we refer to [3].

Let us describe the algorithm for computing of coefficients $(d_l x, \gamma)$. Suppose that the multiplicity of the orbit γ is equal to k . Then the coefficient $(d_l x, \gamma)$ counts rigid configuration

$$(g, u, c) , \quad (14)$$

where

- g is a minus gradient trajectory of the function f_U which begins at x and ends at a point $p \in \partial U$; note that ∂U can be canonically identified with the S^1 -bundle associated with the complex line bundle ν , and thus p determines a ray l_p in one of the fibers of ν ;
- $u : P_{a,b} \rightarrow W_{middle}$ a holomorphic map such that u exits (resp. enters) some orbits at all positive (resp. negative) punctures, enters an orbit $\tilde{\gamma}$ of multiplicity k at ∞ and $\lim_{z \rightarrow 0} u(z) \in l_p$; note that the set of positive punctures must be non-empty due to the maximum principle;
- c is a minus gradient trajectory of the function f_k connecting $\tilde{\gamma}$ and γ .

A new feature of configurations (g, u, c) considered above is the ray l_p which connects the holomorphic curve u with the gradient trajectory g . This requires a justification which will be given elsewhere.

This completes the description of the differential d on QB .

COMPARISON OF FLOER AND BOURGEOIS-OANCEA HOMOLOGIES:

Recall that D denotes the subspace of C consisting of elements of action < 1 . Denote by $QD \subset QB$ the subspace consisting of elements of action < 1 . We shall use the following equality:

$$H_2(C/D, \delta) = H_2(QB/QD, d) . \quad (15)$$

Note that for $i = 1, 2, 3$ we have $C_i = QB_i = \oplus_s B_{i+4s} \otimes_{\mathbb{C}} t^{-s}$, and $D_i = QD_i = QB_i \cap D$. Let us compare the differentials. The stretching-the-neck procedure described above has the following effect on the original Floer trajectories of our Hamiltonian H_E : every Floer trajectory joining a pair of upper generators γ_+ and γ_- splits into three pieces. The piece lying in $W_{middle} = \nu \setminus \Sigma$ is the Floer trajectory joining γ_+ and γ_- with positive punctures (corresponding to the intersection points with Σ) and possibly some negative punctures. The orbit γ_+ lies on the connected component of the ideal boundary of W_{middle} adjacent to W_{right} , while the orbit γ_- lies on the connected component of the ideal boundary of W_{middle} adjacent to W_{left} . The positive punctures are capped by rigid holomorphic planes (the fibers of ν) lying in W_{right} . The negative punctures are capped by rigid holomorphic planes lying in W_{left} corresponding to $\mathbb{CP}^1 \times \text{point}$ and $\text{point} \times \mathbb{CP}^1$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus every negative puncture is capped by exactly 2 rigid planes, which yields the factor 2^{l_-} in the definition of the weight in (10).

Let's focus on the piece lying in W_{middle} : one extends Bourgeois-Oancea theory [5, Prop. 4] and finds an isomorphism between homology whose differential is determined by such punctured Floer trajectories and the homology whose differential is described by configurations of the form (12) and (13). This explains equality (15). The formal proof will be given in a forthcoming paper.

Remark 3.1. From the viewpoint of the Hamiltonian Floer theory, the differential $\delta : C/D \rightarrow C/D$ does not lift canonically to a differential $C \rightarrow C$: the square of the expression in the right hand side of formula (6) vanishes modulo the subcomplex D . Since $d^2 = 0$, the argument above shows that in the limit, “when the neck is stretched”, the square of this expression vanishes in C itself.

UNPERTURBED DIFFERENTIAL:

The explicit form of the differential d_0 is folkloric (private communications of F.Bourgeois and T.Ekholm). We shall present the result right now.

Lemma 3.2. *We have*

$$\begin{aligned}
d_0 \hat{m}_k &= 0, d_0 \hat{M}_k = 0, k \geq 1, \\
d_0 \check{m}_k &= 2\hat{m}_{k-1} + 2\hat{M}_{k-2}, k \geq 3, \\
d_0 \check{M}_k &= 2\hat{m}_k + 2\hat{M}_{k-1}, k \geq 2, \\
d_0 \check{m}_2 &= 2\hat{m}_1 + 2x_2, \\
d_0 \check{M}_1 &= 2\hat{m}_1 + 2x_2, \\
d_0 \check{m}_1 &= 0, \\
d_0 x_0 &= 0, d_0 x_2 = 0.
\end{aligned}$$

It follows that the homology of the complex (B, d_0) are given by

$$\begin{aligned}
H_0(B, d_0) &= \text{Span}_{\mathbb{C}}([x_0]), H_2(B, d_0) = \text{Span}_{\mathbb{C}}([x_2]), \\
H_{2k+2}(B, d_0) &= \text{Span}_{\mathbb{C}}([\hat{M}_k]), k \geq 1, \\
H_1(B, d_0) &= \text{Span}_{\mathbb{C}}([\check{m}_1]), \\
H_{2k+1}(B, d_0) &= \text{Span}_{\mathbb{C}}([\check{M}_k - \check{m}_{k+1}]), k \geq 1.
\end{aligned}$$

It remains to describe the differential d_l for $l \geq 1$.

CALCULATION OF THE QUANTUM CORRECTIONS:

Lemma 3.3. (i) *For all $l \geq 2$ one has $d_l = 0$;*

(ii) *For all $k \geq 1$,*

$$\begin{aligned}
d_1 x_0 &= d_1 x_2 = 0 \\
d_1 \check{m}_k &= \hat{m}_{k+1}, d_1 \check{M}_k = \hat{M}_{k+1}, d_1 \hat{m}_k = 0, d_1 \hat{M}_k = 0.
\end{aligned}$$

Proof of Main Lemma:

1) Define a decreasing filtration $\cdots \supset QB^{(\mu)} \supset QB^{(\mu+1)} \supset \cdots$ on QB by

$$QB_*^{(\mu)} = \oplus_{s \geq \mu+1} C_{*,s} = \oplus_{s \geq \mu+1} t^{-s} B_{*+4s} ,$$

and observe that the differential d preserves the filtration. Furthermore, by Lemma 2.1(iv) for E large enough the subspace $QD_i \subset QB_i$, $i = 1, 2, 3$ consisting of elements of the H_E -action < 1 is squeezed between $QB_i^{(\mu+)}$ and $QB_i^{(\mu-)}$ with $\mu_{\pm}(E) \rightarrow \infty$ as $E \rightarrow \infty$:

$$QB_i^{(\mu-)} \supset QD_i \supset QB_i^{(\mu+)} .$$

We shall show in the next step that for μ large enough $H_2(QB/QB^{(\mu)}, d)$ is isomorphic to \mathbb{C} and is generated by $x_2 \bmod QB^{(\mu)}$. The latter implies that the map $H_2(QB/QB^{(\mu+)}, d) \rightarrow H_2(QB/QB^{(\mu-)}, d)$ is an isomorphism. Since it factors through $H_2(QB/QD, d)$ we shall conclude that $H_2(QB/QD, d) = \mathbb{C}$.

2) Lemma 3.3 yields that $d_1 : H_{2k}(B, d_0) \rightarrow H_{2k+3}(B, d_0)$ vanishes while $d_1 : H_{2k+1}(B, d_0) \rightarrow H_{2k+4}(B, d_0)$ is onto for all $k \geq 0$. Fix μ large enough and identify $\overline{QB} := QB/QB^{(\mu)}$ with $\oplus_{s \leq \mu} C_{*,s}$.

We claim that

$$H_2(\overline{QB}, d) = H_2(B, d_0) = \mathbb{C} .$$

Indeed, consider a filtration $\mathcal{F}_p \overline{QB}_* := \oplus_{\mu \geq s \geq \mu-p} C_{*,s}$ on the complex \overline{QB} . Look at the homology spectral sequence corresponding to this filtration [19]: we have that

$$E_j^1 = \oplus_{s \leq \mu} H_j(C_{*,s}, d_0) = \oplus_{s \leq \mu} t^{-s} H_{j+4s}(B, d_0)$$

and the differential $d^1 : E_j^1 \rightarrow E_{j-1}^1$ can be written as

$$d^1 = \oplus d_s^1, \quad d_s^1 = [t^{-1} d_1] : H_j(C_{*,s}, d_0) \rightarrow H_{j-1}(C_{*,s+1}, d_0), \quad s \leq \mu - 1 .$$

Since $d = d_0 + t^{-1} d_1$, the spectral sequence degenerates at the second page and converges to $E_j^2 := H_j(E^1, d^1)$, and in particular $H_2(\overline{QB}, d) = E_2^2$. In order to calculate E_2^2 look at the piece

$$H_3(C_{*,s-1}, d_0) \rightarrow H_2(C_{*,s}, d_0) \rightarrow H_1(C_{*,s+1}, d_0) \quad (16)$$

of the complex (E^1, d^1) . If $\mu - 1 \geq s \geq 1$, the left arrow is onto, while the right arrow is zero, thus the homology vanishes. If $s = \mu$, the sequence (16) degenerates to

$$H_3(C_{*,\mu-1}, d_0) \rightarrow H_2(C_{*,\mu}, d_0) \rightarrow 0 ,$$

and since the left arrow is onto, the homology vanishes. Finally, for $s = 1$ the sequence (16) degenerates to

$$0 \rightarrow H_2(C_{*,0}, d_0) \rightarrow H_1(C_{*,1}, d_0) .$$

Since the right arrow vanishes, the resulting homology is

$$H_2(C_{*,0}, d_0) = H_2(B, d_0) = \mathbb{C} .$$

We conclude that $H_2(\overline{QB}, d) = E_2^2 = \mathbb{C}$, and the generator of $H_2(\overline{QB}, d)$ is $x_2 \bmod QB^{(\mu)}$, as required.

It remains to put all the pieces together: The calculation above together with equalities (7) and (15) yield

$$HF_2^{(1;E+1)}(H_E) = H_2(C/D, \delta) = H_2(QB/QD, d) = H_2(B, d_0) = \mathbb{C} .$$

This completes the proof. \square

Proof of Lemma 3.3:

1) First we explore coefficients of the form $(d_l \gamma_+, \gamma_-)$ with $l = l_+ \geq 1$,

$$\gamma_{\pm} \in \{\check{m}_{k_{\pm}}, \hat{m}_{k_{\pm}}, \check{M}_{k_{\pm}}, \hat{M}_{k_{\pm}}\} .$$

The index formula reads

$$CZ(\gamma_+) - CZ(\gamma_-) + 4l_+ = 1 .$$

Recall that

$$CZ(\gamma_{\pm}) = 2k_{\pm} + j_{\pm}, \quad j_{\pm} = -1, 0, 1, 2 .$$

Put $h = j_+ - j_-$. Thus the index formula yields

$$2(k_+ - k_-) + h + 4l_+ = 1 . \tag{17}$$

Consider a configuration of the form

$$\gamma_+ \rightsquigarrow^v \alpha \xrightarrow{u} \beta \rightsquigarrow^w \gamma_- .$$

Denote by Δ the degree of the projection of u to Σ . Since the Chern class of ν equals 2 we have

$$(k_+ + l_+) - (k_- + l_-) = 2\Delta . \tag{18}$$

It follows from (17) and (18) that

$$l_+ + l_- = (1 - h)/2 - 2\Delta .$$

Since $l_+ \geq 1$, $\Delta \geq 0$ and h is an integer from $[-3; 3]$ we conclude that $\Delta = 0$. Thus our holomorphic curve lies in the single fiber of the bundle ν .

We claim that $h \neq -3$. Indeed otherwise we have the connecting trajectory of the form

$$\check{m}_{k_+} \xrightarrow{v} \alpha \xrightarrow{u} \beta \xrightarrow{w} \hat{M}_{k_-} . \quad (19)$$

Since \check{m}_{k_+} is the point of minimum of f_{k_+} , v is the constant trajectory. Since \hat{M}_{k_-} is the point of maximum of f_{k_-} , w is the constant trajectory. Since u lies in the single fiber, the set of limit points of $u(z)/|u(z)|$ as $z \rightarrow \infty$ lie on a circle which is the fiber of $Z_{k_-} = \mathbb{R}P^3$ over $\pi^{k_+}(\check{m}_{k_+})$. Generically, as the dimension count shows, this circle does not pass through \hat{M}_{k_-} , and hence configuration (19) does not exist. The claim follows.

Since $h \neq -3$, we get that

$$h = -1, l_+ = 1, l_- = 0 .$$

This readily yields that $d_l \gamma_+ = 0$ for $l \geq 2$ and the only possible non-trivial matrix coefficients could be (with $k_+ := k$) $(d_1 \check{m}_k, \hat{m}_{k+1})$, $(d_1 \check{M}_k, \hat{M}_{k+1})$ and $(d_1 \hat{m}_k, \check{M}_{k+1})$. We claim that

$$(d_1 \check{m}_k, \hat{m}_{k+1}) = (d_1 \check{M}_k, \hat{M}_{k+1}) = 1 .$$

Let us present a calculation (modulo orientations) for the coefficient $(d_1 \check{m}_k, \hat{m}_{k+1})$ (the calculation for $(d_1 \check{M}_k, \hat{M}_{k+1})$ is analogous). Since $\Delta = 0$ and $l_+ = 1$, we work with holomorphic maps u of the cylinder $P_{\{b\}}$ punctured at a point $b \in \mathbb{C} \setminus 0$ lying in the single fiber of ν . Choose functions f_k and f_{k+1} on Z_k and Z_{k+1} respectively so that

$$\pi^k(\check{m}_k) = \pi^{k+1}(\hat{m}_{k+1}) := m \in \Sigma$$

and so that the circle $(\pi^{k+1})^{-1}(m) \subset Z_{k+1}$ forms the unstable manifold \mathcal{U} of \hat{m}_{k+1} . Identify the fiber of the line bundle ν over m with \mathbb{C} . Recall that we identified each Z_j with the unit circle bundle of ν . Assume that \check{m}_k corresponds to the point $-1 \in \mathbb{C}$ and \hat{m}_{k+1} corresponds to the point $1 \in \mathbb{C}$. Since \check{m}_k is the minimum point of f_k , the only gradient trajectories exiting \check{m}_k are the constant ones. Thus we are counting configurations of the form

$$\check{m}_k \xrightarrow{u} \beta \xrightarrow{w} \hat{m}_{k+1} . \quad (20)$$

Note that the point A_β lies both on the circle $\mathcal{U} = (\pi^{k+1})^{-1}(m)$ (since the image of u is contained in the single fiber) and on the stable manifold of \hat{m}_{k+1} . These two submanifolds intersect transversally at a single point, \hat{m}_{k+1} . In particular, w is constant. Therefore it suffices to show that the holomorphic map u is unique up to a reparameterization and up to multiplication by constants (these symmetries should be taken into account when one passes to the moduli space).

The general form of u is $u(z) = \lambda z^k(z - b)$ with the asymptotic conditions

$$\text{Arg}(u(t)) \rightarrow 0, \quad t \in \mathbb{R}_+, t \rightarrow +\infty$$

and

$$\text{Arg}(u(t)) \rightarrow \pi, \quad t \in \mathbb{R}_+, t \rightarrow 0.$$

This readily yields $\lambda, b \in \mathbb{R}_+$. The change of variables $z \rightarrow cz$ with $c \in \mathbb{R}_+$ takes u to the form $u(z) = \lambda c^{k+1} z^k(z - b/c)$. Thus u , up to a reparameterization and up to multiplication by constants coincides with $z^k(z - 1)$. Thus we have the unique configuration of the form (20), which completes the calculation.

Finally, we claim that $(d_1 \hat{m}_k, \check{M}_{k+1}) = 0$. Indeed, assume that $d_1 \hat{m}_k = n \check{M}_{k+1}$. Observe that since $d^2 = 0$ we have that $d_1 d_0 + d_0 d_1 = 0$. Thus (in view of the explicit formulas for the unperturbed differential)

$$d_1 d_0 \hat{m}_k + d_0 d_1 \hat{m}_k = 0 + n d_0 \check{M}_{k+1} = 0.$$

Since $d_0 \check{M}_{k+1} \neq 0$ we conclude that $n = 0$ as claimed.

2) Now we turn to calculation of $d_l x$ where $x \in \{x_0, x_2\}$ and $l \geq 1$. Let us observe that if $(d_l x, \gamma) \neq 0$ then the orbit γ must have an odd grading, and hence $\gamma = \check{m}_k$, or $\gamma = \check{M}_k$. But in this case if there exists a gradient trajectory c connecting an orbit $\tilde{\gamma}$ with γ , and if $\tilde{\gamma}_\alpha$ differs from $\tilde{\gamma}$ by a reparameterization $s \mapsto s e^{i\alpha}$, then for almost all values α there exists a gradient trajectory c_α connecting $\tilde{\gamma}_\alpha$ with γ . This implies that there are no rigid configurations (g, u, c) (see (14) above) which may contribute to $(d_l x, \gamma)$, and hence $(d_l x, \gamma) = 0$. Indeed, any such configuration belongs to a family $(g, u \circ r_\alpha, c_\alpha)$, where $r_\alpha : \mathbb{C} \setminus 0 \rightarrow \mathbb{C} \setminus 0$ is the rotation $z \mapsto z e^{i\alpha}$.

This finishes off the proof of the lemma. \square

Outline of the proof of Lemma 3.2: First we discuss coefficients of the form $(d_0 \gamma_+, \gamma_-)$ with $l_+ = 0$,

$$\gamma_\pm \in \{\check{m}_{k\pm}, \hat{m}_{k\pm}, \check{M}_{k\pm}, \hat{M}_{k\pm}\}.$$

We argue as in Step 1 of the proof of Lemma 3.3, keeping the same notations and taking into account that $l_+ = 0$. This yields three possibilities:

- (i) $h = -3, \Delta = 0, l_- = 2$;
- (ii) $h = -3, \Delta = 1, l_- = 0$;
- (iii) $h = -1, \Delta = 0, l_- = 1$.

Case (i) is ruled out exactly as in the proof of Lemma 3.3. Case (ii) yields $(d_0\check{m}_k, \hat{M}_{k-2}) = 2$, case (iii) yields

$$(d_0\check{m}_k, \hat{m}_{k-1}) = (d_0\check{M}_k, \hat{M}_{k-1}) = 2 ,$$

while all other coefficients vanish.

Further, we analyze the matrix coefficients involving x_0 and x_2 . Equality $(d_0\check{m}_2, x_2) = 2$ follows from the count of degree 1 (properly parameterized) sections of the bundle ν passing through a given generic point and having a single zero of order two at the point $\pi^2(\check{m}_2) \in \Sigma$. Equality $(d_0\check{M}_1, x_2) = 2$ corresponds to the fact that there are exactly two rigid spheres $S^2 \times \text{point}$ and $\text{point} \times S^2$ in W passing through x_2 .

The only tricky remaining coefficient is $(d_0\check{m}_1, x_0) = 0$: Seemingly, there are two rigid spheres in W asymptotic to \check{m}_1 which may contribute to this coefficient. We claim that in fact a cancelation happens. To see this, recall that by (9) and [7] $H_1(B, d_0) = \mathbb{Z}$. If $(d_0\check{m}_1, x_0) \neq 0$, we would get that $H_1(B, d_0) = 0$, and thus arrive at a contradiction. \square

Acknowledgements. We thank Luis Diogo, Misha Entov, Sam Lisi, Dusa McDuff, Isidora Milin, Yasha Savelyev and Frol Zapolsky for numerous useful discussions. Preliminary results of this note were announced at the PRIMA congress in Sydney in July 2009. The second named author thanks the Simons Foundation for sponsoring his stay at MSRI, Berkeley, where a part of this paper has been written, and MSRI for hospitality and a stimulating research atmosphere.

References

- [1] Abbondandolo, A., Schwarz, M., *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), 254–316.

- [2] Bourgeois, F., Eliashberg, Y., Hofer, H., Wysocki, K., Zehnder, E., *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.
- [3] Bourgeois, F., Ekholm, T., Eliashberg, Y., *Effect of Legendrian Surgery*, preprint arXiv:0911.0026, 2009.
- [4] Biran, P., Cornea, O., *Rigidity and uniruling for Lagrangian submanifolds*, Geom. Topol. **13** (2009), 2881–2989.
- [5] Bourgeois, F., Oancea, A., *An exact sequence for contact and symplectic homology*, Invent. Math. **175** (2009), no. 3, 611–680.
- [6] Chekanov, Y., Schlenk, F., *Notes on monotone Lagrangian twist tori*, preprint arXiv:1003.5960, 2010.
- [7] Cohen, R., Jones, J., Yan, J., *The loop homology algebra of spheres and projective spaces*, in *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001)*, 77–92, Progr. Math., 215, Birkhäuser, Basel, 2004.
- [8] Eliashberg, Y., Givental, A., Hofer, H., *Introduction to symplectic field theory*, Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.
- [9] Eliashberg, Y., Kim, S.-S., Polterovich, L., *Geometry of contact transformations and domains: orderability versus squeezing*, Geom. Topol. **10** (2006) 1635–1747.
- [10] Entov, M., Polterovich, L., *Calabi quasimorphism and quantum homology*, Intern. Math. Res. Notices **30** (2003), 1635–1676.
- [11] Entov, M., Polterovich, L., *Quasi-states and symplectic intersections*, Comm. Math. Helv. **81**:1 (2006), 75–99.
- [12] Entov, M., Polterovich, L., *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826.
- [13] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K., *Toric degeneration and non-displaceable Lagrangian tori in $S^2 \times S^2$* , preprint arXiv:1002.1660, 2010.
- [14] Lerman, E., *Symplectic cuts*, Mathematical Research Letters **2** (1995), 247–258.

- [15] McDuff, D., *Displacing Lagrangian toric fibers via probes*, preprint arXiv:0904.1686, 2009.
- [16] McDuff, D., Salamon, D., *J-holomorphic curves and symplectic topology*, AMS, 2004.
- [17] Oh, Y.-G., *Construction of spectral invariants of Hamiltonian diffeomorphisms on general symplectic manifolds*, in *The breadth of symplectic and Poisson geometry*, 525–570, Birkhäuser, Boston, 2005.
- [18] Pelayo, A., Vũ Ngọc, S., *Semitoric integrable systems on symplectic 4-manifolds*, *Invent. Math.* **177** (2009), 571–597.
- [19] Spanier, E.H., *Algebraic topology*, Springer-Verlag, New York-Berlin, 1981.
- [20] Salamon, D. A., Weber, J., *Floer homology and the heat flow*, *Geom. Funct. Anal.* **16** (2006), 1050–1138.

<p>Yakov Eliashberg Stanford University eliash@math.stanford.edu</p>	<p>Leonid Polterovich University of Chicago and Tel Aviv University polterov@runbox.com</p>
--	---